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Quartic quantum theory: an extension of the standard quantum mechanics

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Abstract

We propose an extended quantum theory, in which the number K of parameters necessary to characterize a quantum state behaves as *fourth* power of the number N of distinguishable states. As the simplex of classical N-point probability distributions can be embedded inside a higher-dimensional convex body $\mathcal{M}_N^Q$ of mixed quantum states, one can further increase the dimensionality constructing the set of extended quantum states. The embedding proposed corresponds to an assumption that the physical system described in the N-dimensional Hilbert space is coupled with an auxiliary subsystem of the same dimensionality. The extended theory works for simple quantum systems and is shown to be a non-trivial generalization of the standard quantum theory for which $K = N^2$. Imposing certain restrictions on initial conditions and dynamics allowed in the quartic theory one obtains quadratic theory as a special case. By imposing even stronger constraints one arrives at the classical theory, for which $K = N$.

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1. Introduction

For a long time quantum mechanics (QM) has been one of the most important cornerstones of modern physics. Although predictions of quantum theory were not found to be in contradiction with results of physical experiments, there exist many reasons to look for possible generalizations of quantum mechanics, see, e.g. [1–5].

One possible way to tackle the problem is to follow an axiomatic approach to quantum mechanics and to study consequences of relaxing some of the axioms. An axiomatic approach to quantum theory was initiated by Mackey [6], Ludwig [7, 8] and Piron [9] several decades ago and further developed in several more recent contributions [10–13]. In an influential work of Hardy it is shown that quantum mechanics is a kind of probability theory for which the set of pure states is continuous [10]. This contrasts with the classical probability theory, for which pure states form a discrete set of corners of the probability simplex.

Restricting attention to the problem of finite number of N distinguishable states and analyzing composite systems it is possible to conclude that the number K of degrees of freedom (i.e. the number of parameters required to specify a given state) satisfies the relation $K = N^r$ with an integer exponent. The linear case, $r = 1$, gives the classical probability theory. The quadratic case leads to the standard quantum theory, for which it is necessary to use $K = N^2$ real parameters to characterize completely an unnormalized quantum state. However, higher order theories may exist, which include QM as a special case [14]. To single out the standard quantum theory Hardy uses a 'simplicity axiom' and requires that the exponent takes the minimal value consistent with other axioms, which implies $r = 2$.

Vaguely speaking the exponent r counts the number of indices decorating a mathematical object called *state*, used to determine probabilities associated with the outcomes of a measurement. In the classical theory one deals with probability vectors p_i with a single index, while quantum states are described by density matrices ρ_{ij}. Do we need to work with some more complicated objects, like tensors or multi-index density matrices τ_{ijk} or σ_{ijkl}?

The main aim of this work is to propose a higher order theory, which includes QM as its special case. Instead of working with higher order tensors, the theory of which is well developed [15, 16], we remain within the known formalism of standard complex quantum mechanics, and construct an extended quartic (biquadratic) theory, for which $K = N^4$, assuming a coupling with an auxiliary subsystem. The extended quantum mechanics (XM), reduces to the standard quantum theory in a special case of uncoupled auxiliary systems. On the other hand, XM is shown to be a non-trivial generalization of QM. Throughout the entire work only non-relativistic version of quantum theory will be considered. For simplicity we analyze the case of finite-dimensional Hilbert space. Furthermore, we restrict our attention to single-particle systems only.

Well-known effects of decoherence reduce the magnitude of quantum effects and cause a quantum system to behave classically. In a similar way one can introduce analogous effects of 'hyper-decoherence' which cause a system described in the framework of the extended theory to lose its subtle properties and behave according to predictions of standard quantum theory.

It is worth emphasizing that our approach differs from the theory of Adler based on higher order correlation tensors [17], the 'two-state vector formalism' of Aharonov and Vaidman [18], the 'time-symmetric quantum mechanics' of Wharton [19], and the quaternionic version of quantum theory [20–22], which is not consistent with the power like scaling, $K = N^r$, in analogy to the quantum theory of real density matrices.

Higher order theory proposed here is also different from the algebraic approach of Uhlmann who discusses spaces of states constructed from Jordan algebra [23], and from the generalized quantum mechanics developed by Sorkin [24]. Furthermore, the extended theory by construction belongs to the class of *probabilistic* theories (see, e.g. [25]), so it is different from the theory of *hidden variables*, which would allow one to predict an outcome of an individual experiment.

Our approach explores the geometric structures in quantum mechanics and in particular the convexity of the set of quantum states. Such a description of quantum mechanics goes back to classical papers of Ludwig [7] and Mielnik [26, 27] and was reviewed and updated in [28].

The extended quantum theory constructed here is related to the generalized quantum mechanics of Mielnik [29], in which higher order forms on the Hilbert space are considered and methods of constructing nonlinear variants of quantum mechanics are discussed. On the other hand the theory analyzed here is linear, and the sets of extended states and extended measurements are precisely defined.

The paper is organized as follows. Section 2 contains a geometric review of the standard set-up of quantum mechanics in which we describe the sets of quantum states and quantum maps. Discussion of the extended, quartic theory is based on a definition of the N^4-dimensional convex set of extended quantum states, introduced in section 3. In section 4 we describe the set of generalized measurement operations admissible in the extended theory while section 5 concerns the corresponding discrete dynamics. Section 6 contains the evidence that the extended theory forms a non-trivial generalization of the standard, quadratic quantum theory. The possibility of generalizing the quantum theory even further and working with higher order theories is discussed in section 7. The work is concluded with a discussion in section 8, while some information on duality between convex sets is presented in appendix A.

2. Standard quantum theory: quadratic

In this section we review the standard quantum theory and present requisites necessary for its generalization. Quantum mechanics is a probabilistic theory. Probabilities associated with outcomes of a measurement are characterized by a quantum state described by a density operator ρ which acts on the N-dimensional Hilbert space $\mathcal{H}_N$. In this work we shall assume that N is finite. The density operator is Hermitian and positive.

The set of normalized quantum states of size N for which $Tr\,\rho = 1$ will be denoted by $\mathcal{M}_N^Q$. In the simplest case of $N = 2$ the set of mixed states of a single qubit forms the *Bloch ball*, $\mathcal{M}_2^Q = B_3 \subset \mathbb{R}^3$. Degree of mixing of a state ρ can be characterized by the von Neumann entropy, $S(\rho) = -\mathrm{Tr}\,\rho \ln \rho$. This quantity varies from zero for pure states, to $\ln N$ for the maximally mixed state, $\rho_* = \mathbb{1}/N$, located in the center of the set $\mathcal{M}_N^Q$.

To introduce a partial order into the set of mixed states one uses the majorization relation [30]. A density matrix ρ of size N is majorized by a state ω, written $\rho \prec \omega$, if their decreasingly ordered spectra $\vec{\lambda}$ and $\vec{\kappa}$ satisfy: $\sum_{i=1}^m \lambda_i \leqslant \sum_{i=1}^m \kappa_i$, for $m = 1, 2, \ldots, N-1$. The majorization relation implies an inequality between entropies: if $\rho \prec \omega$ then $S(\rho) \geqslant S(\omega)$. Any mixed state ρ satisfies relations $\rho_* \prec \rho \prec |\psi\rangle\langle\psi|$, where $|\psi\rangle \in \mathcal{H}_N$ denotes an arbitrary pure state, see, e.g. [28].

For our purposes it is also convenient to work with *subnormalized states*, such that $\mathrm{Tr}\,\rho \leqslant 1$. The N^2-dimensional set of subnormalized states forms a convex hull of the set of normalized states and the zero state, $\widetilde{\mathcal{M}}_N^Q = \text{conv hull}\big\{\mathcal{M}_N^Q, 0\big\}$.

A one-step linear dynamics in $\mathcal{M}_N^Q$ may be represented in its Kraus form

$$\rho \;\rightarrow\; \rho' = \Phi(\rho) \;=\; \sum_{i=1}^{k} X_i \rho X_i^\dagger, \tag{1}$$

in which the number k of Kraus operators can be arbitrary. Such a form ensures that the map Φ is *completely positive* (CP), which means that an extended map, $\Phi \otimes \mathbb{1}_M$, sends the set of positive operators into itself for all possible dimensions M of the ancilla [31]. The Kraus operators X_i can be interpreted as measurement operators, and the above form provides a way to describe quantum measurement performed on the state ρ: the ith outcome occurs with the probability $p_i = \mathrm{Tr}\, X_i \rho X_i^\dagger$ and the measurement process transforms the initial state according to

$$\rho \;\rightarrow \rho_i = \frac{X_i \rho X_i^\dagger}{\mathrm{Tr}\, X_i \rho X_i^\dagger}. \tag{2}$$

To assure that the trace of ρ does not grow under the action of Φ, the Kraus operators X_i need to satisfy the following inequality [30],

$$\sum_{i=1}^{k} X_i^\dagger X_i \leqslant \mathbb{1}_N. \tag{3}$$

Usage of subnormalized states and trace non-increasing maps corresponds to a realistic physical assumption that the experimental apparatus fails to work with a certain probability and no measurement results are recorded.

The measurement process can be characterized by the elements of POVM (positive operator valued measures), $E_i := X_i^\dagger X_i$. By construction these operators are Hermitian and positive. Due to (3) they fulfil the relation $\sum_i E_i \leqslant \mathbb{1}_N$ hence each individual element satisfies $E_i \leqslant \mathbb{1}_N$. Thus the set of the elements of a POVM in the standard quantum theory can be defined as

$$\mathcal{E}_N^Q := \left\{E_i = E_i^\dagger : E_i \geqslant 0 \text{ and } E_i \leqslant \mathbb{1}_N\right\}. \tag{4}$$

Since the elements of POVM are positive operators, the probability p_i is non-negative,

$$p_i = \operatorname{Tr} \rho E_i \geqslant 0 \qquad \text{for any} \quad \rho \geqslant 0. \tag{5}$$

The above relation shows that the cone containing the elements of POVM is *dual* to the set of subnormalized states, $\mathcal{E}_N^Q = \left(\widetilde{\mathcal{M}}_N^Q\right)^*$. In the case of the standard quantum theory we work with the set of positive operators which is self-dual, so both cones are equal, $\mathcal{E}_N^Q = \widetilde{\mathcal{M}}_N^Q$ (see figure 4(*a*)). For more information on dual cones consult appendix A.

In the special case of equality in (3) the completeness relation $\sum_i X_i^\dagger X_i = \mathbb{1}_N$ imposes that $\operatorname{Tr} \Phi(\rho) = \operatorname{Tr} \rho$. A completely positive trace preserving map is called *quantum operation* or *stochastic* map. If the dual relation is satisfied, $\sum_i X_i X_i^\dagger = \mathbb{1}_N$, the map is called *unital*, since it preserves the maximally mixed state, $\Phi(\rho_*) = \rho_* = \mathbb{1}/N$. A completely positive trace preserving unital map is called *bistochastic*. We are going to use an important property of these maps reviewed in [28]: any initial state majorizes its image with respect to any bistochastic map, $\Phi_B(\rho) \prec \rho$.

Treating ρ as an element of the Hilbert–Schmidt space of operators, we may think of Φ as a super-operator, (a square matrix of size N^2), acting in this space,

$$\rho'_{m\mu} = \Phi_{\substack{m\mu \\ n\nu}} \rho_{n\nu}, \tag{6}$$

where summation over repeated indices has to be performed. The operators acting on the vectors of the Hilbert–Schmidt space are often called *super-operators*, in order to distinguish them from the operators of the HS space itself. Let us emphasize that this common notion [32] used since the 1960s [33] is not related to supersymmetric theories.

The super-operator Φ can be represented by means of tensor products of the Kraus operators,

$$\Phi = \sum_{i=1}^{k} X_i \otimes \bar{X}_i. \tag{7}$$

The matrix Φ needs not to be Hermitian. However, reshuffling its elements one defines a Hermitian *dynamical matrix*, $D = D(\Phi)$ [34]. Its elements read

$$D_{\substack{mn \\ \mu\nu}} := \Phi_{\substack{m\mu \\ n\nu}} = \left(\Phi_{\substack{mn \\ \mu\nu}}\right)^R. \tag{8}$$

The symbol ^{R} denotes the transformation of reshuffling of elements of a four-index matrix, which exchanges two indices, μ and n in the formula above [28].

A theorem of Choi [35] states that the map Φ is completely positive, if the dynamical matrix is positive, $D(\Phi) \geqslant 0$. Therefore D, also called *Choi* matrix, might be interpreted as a Hermitian operator acting on a composed Hilbert space $\mathcal{H}_{N^2} = \mathcal{H}_A \otimes \mathcal{H}_B$ of size N^2. The trace non-increasing condition (3) is equivalent to the following constraint on the dynamical matrix, $\mathrm{Tr}_A D \leqslant \mathbb{1}_N$. Rescaling the Choi matrices $\sigma = D/N$, we recognize that the set of trace non-increasing maps forms a convex subset of the N^4-dimensional set of subnormalized states on $\mathcal{H}_{N^2}$ [36]. This duality between linear maps and states on the enlarged system is called *Jamiołkowski isomorphism*, which refers to his early contribution [37].

To appreciate this duality let us look at an extended operation $(\Phi \otimes \mathbb{1})$ acting on the maximally entangled state

$$|\psi^+\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle \otimes |i\rangle \tag{9}$$

from the enlarged Hilbert space, $\mathcal{H}_N \otimes \mathcal{H}_N$. The dynamical matrix D corresponding to the map Φ reads then

$$D(\Phi) = N(\Phi \otimes \mathbb{1})|\psi^+\rangle\langle\psi^+|. \tag{10}$$

The state-map isomorphism, written above for states with maximally mixed partial trace, $\mathrm{Tr}_A D = \mathbb{1}_N$, can be generalized also for other states [38].

After reviewing some basic properties of discrete quantum dynamics, let us see how classical dynamics emerges as a special case of the quantum theory. Consider the set of normalized diagonal density matrices, $\rho_{ij} = p_i \delta_{ij}$, which forms the $(N-1)$-dimensional simplex $\mathcal{M}_N^C = \Delta_{N-1}$ of classical probability distributions. If we restrict our attention to maps described by diagonal dynamical matrices, $D_{\substack{m\mu \\ n\nu}} = T_{m\mu}\delta_{mn}\delta_{\mu\nu}$, then the diagonal structure of ρ is preserved, so we recover the classical probability theory. Moreover, for any quantum map Φ we obtain corresponding classical dynamics in the simplex of probability distributions, $p'_m = T_{mn} p_n$, by constructing the transition matrix out of diagonal elements of the dynamical matrix.

Lemma 1. *Let Φ be a linear quantum map acting on $\mathcal{M}_N^Q$. Let T denote a square matrix of size N obtained by reshaping the diagonal elements of the corresponding dynamical matrix, $T_{mn} = \Phi_{\substack{mm \\ nn}}$, (without summation over repeating indices). If Φ is a quantum stochastic (bistochastic) map, then T is a stochastic (bistochastic) matrix.*

Proof. If quantum map Φ is completely positive, the corresponding dynamical matrix is positive definite, so all elements of its diagonal used to assemble T are not negative. If Φ is trace preserving, equality in (3) holds, and implies the relation $\sum_n T_{mn} = 1$ for all $m = 1, \dots, N$. If Φ is unital then the dual relation for partial trace of Φ^R implies that $\sum_m T_{mn} = 1$ for all $n = 1, \dots, N$. Thus quantum stochasticity (bistochasticity) of the map Φ implies the classical property of the transition matrix T. □

Any trace non-increasing map, for which the Kraus operators satisfy relation (3), can be also called *sub-stochastic*, since in the case of diagonal D the classical transition matrix T is sub-stochastic [31].

It is instructive to analyze the geometry of the sets of classical and quantum states [28]. For simplicity we have compared in figure 1 the sets of normalized states for $N = 2$. The interval of classical one-bit states $\mathcal{M}_2^C = [0, 1]$ can be embedded inside the 3D Bloch ball consisting of pure and mixed states of one qubit (see figure 1(*b*)). On the other hand, the

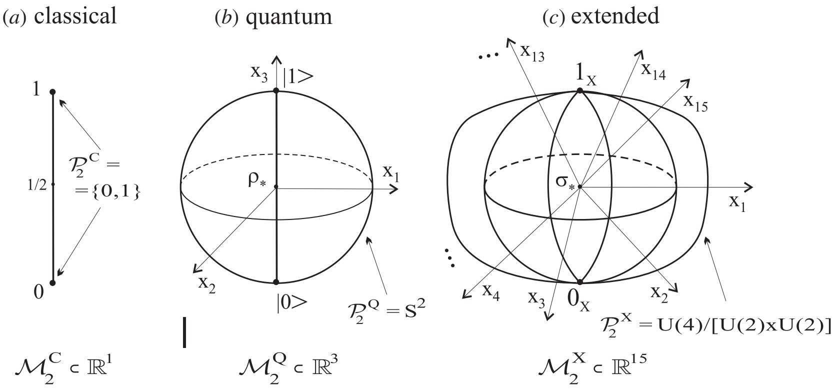


Figure 1. The space of mixed states for $N = 2$: (*a*) classical theory, (*b*) quantum theory, (*c*) extended theory (sketch of a 15-d set).

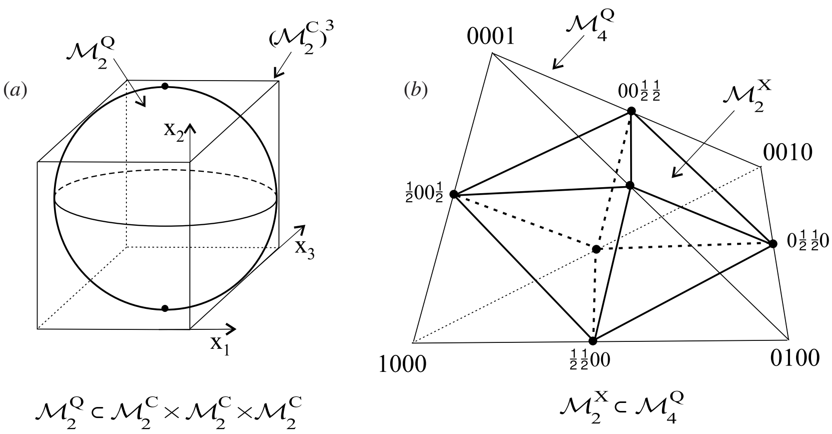


Figure 2. Embedding of (*a*) the Bloch ball of one-qubit states inside the cube of states of three independent bits, (*b*) the octahedron of one-exbit states inside the tetrahedron of two-qubit states (to reduce the dimensionality only spectra are considered).

Bloch ball can be inscribed inside the cube, which describes the set of product states of three classical bits (see figure 2(*a*)). This very cube can be embedded inside the simplex Δ_7 formed by 2^3 corners representing all possible pure states of three classical bits. The cube would describe allowed results of fiducial measurements of three components of the spin $1/2$, if they were independent classical variables [40]. Truncation of the corners of the 3-cube, implied by rules of quantum mechanics, reduces the cube to the ball. After this truncation procedure only two quantum states remain distinguishable. Furthermore, such a qualitative change of the symmetry of the body makes the set of extremal states continuous and allows for an arbitrary rotation of the Bloch ball [14, 39]. Rotating the initial state $|0\rangle$ with respect to any axis perpendicular to the interval of classical states one generates a coherent superposition of $|0\rangle$ and $|1\rangle$. Existence of such a pure state, which does not have a classical analog, explains interference effects, typical of quantum theory.

3. An extended quantum theory: quartic

In order to work out a generalized, quartic theory we need first to define a set of extended states. In this paper we are going to consider mono-partite systems[1] of an arbitrary size N, but to gain some intuition we shall begin with the simplest case of $N = 2$. Analyzing the normalized states of a single *qubit* (quantum bit), we will copy the embedding procedure which blows up the interval $\mathcal{M}_2^C$ of classical states into the Bloch ball $\mathcal{M}_2^Q$ of quantum states. Thus we shall put the 3D Bloch ball of all states of a qubit inside the larger 15D body $\mathcal{M}_2^X$ of an *exbit* (extended bit) as sketched in figure 1(*c*). (In a recent paper of Barrett [40] a similar name of *gbit*, standing for *generalized bit* was introduced.) Designing the shape of the set of extended bits we have to keep in mind that it may contain only two distinguishable states, say 0_X and 1_X.

The dimension of the set $\mathcal{M}_N^X$ of normalized extended states should be equal to $N^4 - 1$, since the remaining dimension is obtained by imposing a weaker condition of subnormalization. Thus it is natural to look at it as a suitable subset of the set $\mathcal{M}_{N^2}^Q$ which contains the mixed states of two *quNits* (systems described in the N-dimensional Hilbert space). As the ball of one-qubit states arises by truncating the corners of the three-bit cube, to define the set of the states of an exbit we propose to reduce the number of distinguishable states in $\mathcal{M}_4^Q$ by truncating the corners of the simplex Δ_3 of eigenvalues of standard quantum states for $N = 4$. In this way one obtains a convex 15-d set of these states, the spectra of which belong to the octahedron formed by 6 centers of edges of the tetrahedron Δ_3 (see figure 2(*b*)).

More formally, let us propose a general definition of the set of extended states for an arbitrary N,

$$\mathcal{M}_N^X := \left\{ \sigma \in \mathcal{M}_{N^2}^Q : \sigma \prec \sigma_0 := |0\rangle\langle 0| \otimes \frac{1}{N}\mathbb{1} \right\}, \tag{11}$$

where $|0\rangle \in \mathcal{H}_N$ represents an arbitrary pure state of the standard theory. The symbol $\prec$ denotes the majorization relation (defined in previous section), with the help of which the truncation is performed. The ordered set of eigenvalues of the extended state σ_0 forms the vector of length N^2 with N non-vanishing components only, $\mathrm{eig}(\sigma_0) = \frac{1}{N}\{1, \ldots, 1, 0, \ldots, 0\} =: \vec{v}$. From a combinatorial point of view the set of all spectra majorized by this vector forms a permutation polytope called *permutohedron* or multipermutohedron [41, 42]. It is defined as a convex hull of all permutations of a given vector, $\mathrm{Perm}(\vec{v}) := \text{conv hull}(P(\vec{v}))$, where the convex hull contains all $k!$ permutations P in the k-element set of components of $\vec{v}$. In the case considered here the vector $\vec{v}$ has $k = N^2$ components, but only N of them are nonzero. Thus the number of corners of $\mathrm{Perm}_N := \mathrm{Perm}(\vec{v})$ is given by the binomial symbol, $C_N = \binom{N^2}{N}$. In the simplest case of $N = 2$ we get $C_2 = 6$ and the set Perm_2 forms a regular octahedron shown in figure 2(*b*). Thus an operator σ belongs to the set of extended states if its spectrum belongs to the permutohedron,

$$\mathcal{M}_N^X = \{\sigma = \sigma^\dagger : \mathrm{eig}(\sigma) \in \mathrm{Perm}_N\}. \tag{12}$$

To show that the described choice of the set $\mathcal{M}_N^X$ is acceptable we need to discuss the number of distinguishable states it supports.

Lemma 2. *The set $\mathcal{M}_N^X$ contains exactly N mutually distinguishable states.*

Proof. Let $\{|i\rangle\}_{i=1}^N$ represent an arbitrary orthonormal basis in $\mathcal{H}_N$. Then extended states $\sigma_i = |i\rangle\langle i| \otimes \mathbb{1}/N$ have non-overlapping supports and can be deterministically discriminated.

[1] Mono-partite systems consist of a single particle only, while bi-partite systems consist of two well-defined subsystems.

To show that $\mathcal{M}_N^X$ does not contain more distinguishable states it is sufficient to apply a lemma, that the sum of the ranks of all distinguishable states is not larger than the total dimension d_t of the Hilbert space [43]. In the case analyzed $d_t = N^2$ and the set $\mathcal{M}_N^X$ does not contain any states with rank smaller than N, so the maximal number of distinguishable states is equal to N. □

By means of a suitable mixture of unitary transformations one can send the state σ_1 into any state σ such that $\sigma \prec \sigma_1$. A convex mixture of unitaries is bistochastic, so any initial state majorizes its image obtained with this map. Thus both conditions are equivalent and we are in a position to formulate an alternative definition of the set of extended states,

$$\mathcal{M}_N^X = \text{conv hull}\left[U\left(|0\rangle\langle 0| \otimes \frac{1}{N}\mathbb{1}\right)U^\dagger\right], \tag{13}$$

where $|0\rangle$ is an arbitrary state in $\mathcal{H}_N$, while U denotes a unitary matrix from $U(N^2)$.

Let us note that the set $\mathcal{M}_N^X$ of extended states defined in equivalent forms (11), (12) and (13) is determined as the minimal set in $N^4 - 1$ dimensions which is invariant under unitary transformations and supports exactly N distinguishable states.

In the simplest case of $N = 2$ the set $\mathcal{M}_2^X$ of extended states has an appealing property. By construction it forms a convex subset of the set of two-qubit states, which contains two sets of separable and entangled states. Consider a three-dimensional subset of the set of two-qubit states defined as convex combination of four Bell states, $|\psi_\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ and $|\phi_\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$. Then the octahedron contained in $\mathcal{M}_2^X$ consists of separable states only, while all other states of the tetrahedron are entangled.

For $N = 2$ any pure state of the extended theory is represented by an $N = 4$ mixed state of the standard theory $\sigma_\phi = |\phi\rangle\langle\phi| \otimes \rho_*$ with spectrum $\{1/2, 1/2, 0, 0\}$. The orbit of pure states of the extended theory contains unitarily equivalent states, which form an eight-dimensional flag manifold, $\mathcal{P}_2^X = U(4)/[(U(2) \times U(2)]$ in contrast to the two-dimensional Bloch sphere, $\mathcal{P}_2^Q = U(2)/[(U(1) \times U(1)] = \mathbb{C}P^1 = S^2$. In general, the set of pure states of the extended theory, $\mathcal{P}_N^X = U(N^2)/[(U(N^2 - N) \times U(N)]$, has $2N^2(N-1)$ dimensions. This is exactly N^2 times more than its quantum counterpart, the complex projective space of $2(N-1)$ real dimensions, $\mathcal{P}_N^Q = U(N)/[(U(N-1) \times U(1)] = \mathbb{C}P^{N-1}$. By construction the entropy of an extended state belongs to the interval $S(\sigma) \in [\ln N, 2\ln N]$, so it is convenient to define a gauged quantity $S_X := S - \ln N$ which vanishes for extended pure states.

To find out how the Bloch ball is embedded inside $\mathcal{M}_2^X$ consider a family of one qubit states $\rho(a)$ with spectrum $\{a, 1-a\}$. The extension of this family has the form $\sigma(\rho) = \rho \otimes \mathbb{1}/2$. The spectra of these extended states read $\{1/2 - a, 1/2 - a, a, a\}$ and form the vertical diagonal of the octahedron which crosses its center and joins the edges $\{1/2, 1/2, 0, 0\}$ with $\{0, 0, 1/2, 1/2\}$ (see figure 2(*b*)). These points represent the logical states of the extended theory, 0_X and 1_X, equal to $|0\rangle\langle 0| \otimes \rho_*$ and $|1\rangle\langle 1| \otimes \rho_*$, respectively. All other points of the Bloch ball are obtained from points of the interval $[0_X, 1_X]$ by local unitary rotation, $V \otimes \mathbb{1}$. Note that the points of the octahedron $\mathcal{M}_2^X$ beside the vertical interval $[0_X, 1_X]$ do not have one-to-one analogs in the quantum theory.

In the case of an arbitrary N an extension of ρ is obtained by adding an ancilla in the maximally mixed state,

$$\rho \to \sigma \equiv \rho \otimes \frac{1}{N}\mathbb{1}_N \qquad \text{hence} \qquad \sigma_{ijkl} = \frac{1}{N}\rho_{ik}\delta_{jl}. \tag{14}$$

By construction these states belong to $\mathcal{M}_N^X$ and act on an extended Hilbert space $\mathcal{H}_{N^2} = \mathcal{H}_A \otimes \mathcal{H}_{A'}$. Moreover, pure states of the standard quantum theory with vanishing entropy are mapped into extremal states of the extended theory with entropy equal to $\ln N$.

In general a state σ of the extended theory need not have the product form (14), since the state ρ may be entangled with the ancilliary system. A bipartite state $\sigma \in \mathcal{M}_N^X$ will be called an *extension* of the quantum state $\rho \in \mathcal{M}_N^Q$ if the marginal satisfies

$$\mathrm{Tr}_{A'}(\sigma) = \rho. \tag{15}$$

Reduction by partial trace is not reversible, and a given mixed state ρ may have several different extensions σ, such that $\mathrm{Tr}_{A'}(\sigma) = \rho$. However, any pure state has a unique extension only. It has a tensor product form (14) and reads $\sigma_\phi = |\phi\rangle\langle\phi| \otimes \frac{1}{N}\mathbb{1}$.

Extended states $\sigma \in \mathcal{M}_N^X$ are defined on a bipartite system and can be interpreted in view of the Jamiołkowski isomorphism (10): a state σ of the extended theory may be considered as a completely positive quantum map acting on $\mathcal{M}_N^Q$ and determined by $D = N\sigma$. In general, a map need not be trace preserving, since this is only true if $\mathrm{Tr}_A\sigma = \mathbb{1}/N$.

In particular this is the case for all product extensions (14) for which $D = \rho \otimes \mathbb{1}$. The corresponding map Φ_ρ acts as a complete one-step contraction, and sends any initial state ω into ρ,

$$\Phi_\rho(\omega) = \rho \qquad \text{for any} \quad \omega \in \mathcal{M}_N^Q. \tag{16}$$

To show this let us start with the dynamical matrix of this map, $D_{\substack{mn\\ \mu\nu}} = \rho_{m\mu}\delta_{n\nu}$. Writing down the elements of the image $\omega' = \Phi_\rho(\omega) = D^R\omega$ in the standard basis we obtain the desired result, $\omega'_{m\mu} = D_{\substack{mn\\ \mu\nu}}\omega_{n\nu} = \rho_{m\mu}(\mathrm{Tr}\,\omega) = \rho_{m\mu}$.

Consider now an arbitrary state $\sigma \in \mathcal{M}_N^X$, prepared as an extension of $\rho = \mathrm{Tr}_B\sigma$. This relation can be rewritten with the help of the super-operator (7), dynamical matrix (8) and Jamiołkowski isomorphism (10),

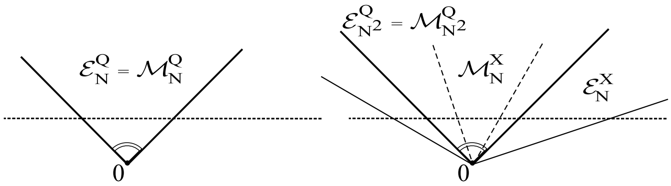
$$\rho = \mathrm{Tr}_B\sigma = \frac{1}{N}\sum_{i=1}^{k} X_i X_i^\dagger = \Phi(\mathbb{1}/N). \tag{17}$$

In this way we have arrived at a dynamical interpretation of objects of the extended theory: a state $\rho \in \mathcal{M}_N^Q$ can be extended to $\sigma \in \mathcal{M}_N^X$, which represents a linear map $\Phi : \mathcal{M}_N^Q \to \mathcal{M}_N^Q$, such that its effect is equal to ρ. It means that Φ sends the maximally mixed state $\rho_* = \mathbb{1}/N$ into ρ. In particular the trivial (product) extension (14) represents the complete contraction, which sends every initial state into ρ.

In analogy to the N^2-dimensional set $\widetilde{\mathcal{M}}_N^Q$ of quantum subnormalized states we define subnormalized extended states, satisfying $\mathrm{Tr}\,\sigma \leqslant 1$. The set of all such states, $\widetilde{\mathcal{M}}_N^X = \text{conv hull}\{\mathcal{M}_N^X, 0\}$ has N^4 dimensions, as required. Thus the number K of parameters necessary to characterize a given state σ of the theory behaves as the forth power of N. This property justifies the name used in the title of the paper: the extended theory proposed in this work can be called *quartic*.

Let us compare our construction with the generalized quantum mechanics of Mielnik [29]. In his approach the set of extended states contains 'density tensors' constructed of convex combinations of separable pure states. On the other hand the set of extended states $\mathcal{M}_N^X$ also contains states which are not separable with respect to the fictitious splitting into the 'physical system' and the 'hypothetical ancilliary system'. Such a possible entanglement between the 'system' and the 'ancilla' plays a crucial role in the dynamics: as shown in the subsequent sections it contributes to the fact that the predictions of the standard quadratic theory and the generalized quartic theory can be different.

To summarize this section, a quartic extension of the quadratic quantum theory is constructed by extending the set of admissible states. Any extended state σ can be interpreted as if the corresponding state ρ of the standard theory was entangled with an auxiliary subsystem



(*a*) quantum (*b*) extended theory

Figure 3. Cones of states $\mathcal{M}$ and elements of POVM $\mathcal{E}$: (*a*) self-dual for quantum theory, $\mathcal{E}_N^Q = \mathcal{M}_N^Q$; (*b*) dual in the extended theory for which $\mathcal{E}_N^X \supset \mathcal{E}_{N^2}^Q = \mathcal{M}_{N^2}^Q \supset \mathcal{M}_N^X = (\mathcal{E}_N^X)^*$.

of the same size in such a way that the state σ of the composite system obeys (11) and its marginal is equal to ρ. Note that such a concept of an ancillary subsystem (the presence of which might be difficult to detect) is introduced for a pedagogical purpose only: in the extended theory the system is described by a single density tensor with four indices, so in practice it cannot be divided into a 'physical particle' and an auxiliary 'ghost-like' subsystem.

A state σ of the extended theory represents a completely positive quantum map which moves the center of the body of quantum states into ρ. As the set of classical states forms the set of diagonal density matrices embedded in $\mathcal{M}_N^Q$, the set of quantum states forms a proper subset of $\mathcal{M}_N^X$ containing product states, $\rho \otimes \mathbb{1}/N$.

4. Extended measurements and POVM

In analogy to the standard quantum theory of a measurement process we will assume that the generalized measurement acting on σ is described by the elements E_i^X of an extended POVM. Conservation of probability implies a relation $\sum E_i^X \leqslant 1$ in analogy to the quantum case.

Furthermore, the probabilities p_i of a single outcome have to be non-negative, so we require that

$$p_i = \operatorname{Tr} \sigma E_i^X \geqslant 0 \qquad \text{for any} \quad \sigma \in \mathcal{M}_N^X. \tag{18}$$

This equation defines the set of elements E_i^X of an XPOVM (an extended POVM). The key difference with respect to the quantum condition (5) is that the extended states σ are not only positive, but they belong to the set $\mathcal{M}_N^X$ which arise by the truncation of the set $\mathcal{M}_{N^2}^Q$ of positive operators acting on an extended system. Hence elements E_i^X of an XPOVM (an extended POVM) may not be positive, provided the condition (18) holds for all admissible states. This relation shows that the set of elements of XPOVM belongs to the cone *dual* to the set of extended states,

$$\mathcal{E}_N^X := \left\{E_i^X = \left(E_i^X\right)^\dagger : E_i^X \in \left(\mathcal{M}_N^X\right)^* \text{ and } E_i^X \leqslant \mathbb{1}_{N^2}\right\}. \tag{19}$$

Here $\mathcal{M}^*$ denotes the set dual to $\mathcal{M}$; see appendix A for definition and properties of dual cones and dual sets.

The geometry of these sets is sketched in figure 3. While in the case of standard quantum theory both sets of quantum states and elements of POVM do coincide (panel a), in the extended theory the set $\mathcal{M}_N^X$ of extended states does not contain all positive operators, so the dual set $\mathcal{E}_N^X$ also contains some operators which are not positive. The boundary of the cone of the elements of POVM has to be perpendicular to the opposite boundary of the set of extended states, since the relation (18) bounds the scalar product in the Hilbert–Schmidt space of linear operators. Thus this relation can be interpreted as a condition that the angle between two corresponding vectors is not larger than $\pi/2$.

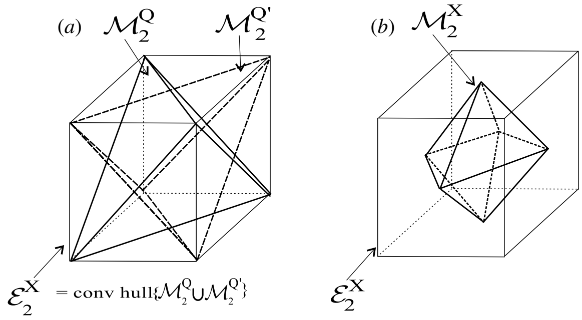


Figure 4. The space of elements of an extended POVM forms (*a*) a convex hull of the set $\mathcal{M}_2^Q$ of states and its image and (*b*) set $\mathcal{E}_2^X$ dual to the set $\mathcal{M}_2^X$ of extended states.

Since the set of extended states is invariant with respect to unitary transformation, its structure is determined by the permutohedron Perm_N containing all admissible spectra. Therefore the set $\mathcal{E}_N^X$ of all elements of XPOVM is unitarily invariant and can be specified by defining its spectra. Lemma 4 proved in appendix A implies that

$$\mathcal{E}_N^X := \left\{E_i^X = \left(E_i^X\right)^\dagger : \mathrm{eig}\left(E_i^X\right) \in (\mathrm{Perm}_N)^* \text{ and } E_i^X \leqslant \mathbb{1}_{N^2}\right\}. \tag{20}$$

Hence to find the set of operators belonging to $\mathcal{E}_N^X$ it is sufficient to find a polytope dual to the permutohedron Perm_N. Each corner of the permutohedron Perm_N generates a face of $(\mathrm{Perm}_N)^*$, so the latter polytope has $C_N = \binom{N^2}{N}$ faces. The structure of both sets is particularly simple in case of $N = 2$, for which one arrives at a pair of dual regular polytopes in $\mathbb{R}^3$: an octahedron and a cube. The spectra of the states from $\mathcal{M}_2^X$ belong to the regular octahedron $\mathrm{Perm}_2 = \mathrm{Perm}(1/2, 1/2, 0, 0)$, so the dual set $\mathcal{E}_2^X$ of elements of XPOVM contains operators with spectra belonging to the cube equal to $(\mathrm{Perm}_2)^*$. The cube can be written as convex hull of a tetrahedron and its mirror copy, $(\mathrm{Perm}_2)^* = \text{conv hull}(\mathrm{Perm}(1, 0, 0, 0) \cup \mathrm{Perm}(1/2, 1/2, 1/2, -1))$ (see figure 4). Observe that $\mathcal{M}_2^X$ also contains non-positive matrices, e.g. a diagonal matrix $\mathrm{diag}(1/2, 1/2, 1/2, -1)$. However, due to duality relation $\mathcal{E}_N^X = \left(\mathcal{M}_N^X\right)^*$, inequality (18) is by construction fulfilled for any extended state $\sigma \in \mathcal{M}_N^X$.

5. Extended dynamics and supermaps

To complete the construction of the quartic theory we have to allow for some action in the set of extended states. As in the case of standard quantum theory we shall discuss only discrete linear maps.

An extended state $\sigma \in \mathcal{M}_N^X$ may be considered as a map on $\mathcal{M}_N^Q$, so we are going to analyze a transformation $\sigma' = \Gamma(\sigma)$ which sends a quantum map into a quantum map. Since in the physics literature a map sending operators into operators is called a 'super-operator', we shall call $\Gamma : \mathcal{M}_N^X \to \mathcal{M}_N^X$ a *supermap*. It is represented by a matrix of size N^4 which acts on an extended state in analogy to (6),

$$\sigma'_{\substack{ab\\cd}} = \sum_{xyzt} \Gamma_{\substack{abcd\\xyzt}} \sigma_{\substack{xy\\zt}}. \tag{21}$$

A supermap corresponds to the concept of *motion*, which transforms the set of extended pure states in the generalized quantum mechanics of Mielnik [29]. Some properties of supermaps were independently investigated in a very recent work by Chiribella, D'Ariano and Perinotti [44].

Investigating linear maps in the set of extended states we aim to accomplish two complementary tasks: (i) for any quantum operation Ψ construct a corresponding supermap $\Gamma = \Gamma(\Psi)$ which preserves the set of quantum states embedded inside the set of extended states. (ii) For any admissible trace-preserving supermap Γ which acts on the set $\mathcal{M}_N^X$ of extended states find a reduced quantum map Φ, which acts the set $\mathcal{M}_N^Q$ and forms a quantum operation, (is completely positive and trace preserving).

Let us first consider the product extension of quantum operations,

$$\rho \to \rho' = \Psi(\rho) \quad \Longrightarrow \quad \sigma \to \sigma' = \Gamma(\sigma) = (\Psi \otimes \mathbb{1})\sigma. \tag{22}$$

If the initial state has the product form, $\sigma = \rho \otimes \mathbb{1}/N$, then $\sigma' = \Psi(\rho) \otimes \mathbb{1}/N$. The maps of form (22) preserve thus the structure of the set of quantum states, and in this way any quantum operation can be realized in the extended set-up. In this way we have arrived at

Proposition 1. *The extended, quartic theory is a generalization of the standard quantum theory. In the special case of the tensor product structure of initial states and supermaps, XM reduces to QM.*

The special case of supermaps of the product form (22) has a simple interpretation in view of the Jamiołkowski isomorphism. Associating by means of (10) initial and final states, σ and σ', with the maps Φ and Φ' we realize that the map $\Gamma = \Psi \otimes \mathbb{1}$ acts in the space of extended states (identified with the set of maps) as a *composition*, $\Phi' = \Psi \cdot \Phi$. Going back into the space of states with Jamiołkowski isomorphism into the space of quantum states one can define in this way a composition of states [45], $\sigma_\Psi \odot \sigma_\Phi := \left(\sigma_\Psi^R \sigma_\Phi^R\right)^R$.

Let us now relax the assumption (22) on the product form and look for a more general class of linear maps. In general we need to work with maps Γ that preserve the set of extended states; if $\sigma \in \mathcal{M}_N^X$ then $\Gamma(\sigma) \in \mathcal{M}_N^X$. This property parallels the positivity of quantum maps which preserve the set $\mathcal{M}_N^Q$. However, analyzing dynamics of a quantum system one takes into account the possible presence of an ancilla and defines completely positive maps. Hence we advance the following notion of *completely preserving* maps, related to the concept of *well-defined* transformations used by Barrett in [40].

Definition. *Consider a given sequence of convex sets Q_k labeled by an integer k and a map Γ defined on Q_N. The map Γ is called preserving if $\Gamma(Q_N) \subset Q_N$. We say that the map Γ is completely preserving if its extension $\Gamma \otimes \mathbb{1}_M$ acting on Q_{NM} is preserving for an arbitrary M.*

Taking for Q_N the set $\mathcal{M}_N^Q$ of quantum states we get back the standard definition of CP maps. However, if we put for Q_N the set $\mathcal{M}_N^X$, we get the characterization of the maps which completely preserve the set of extended states, so are admissible in the quartic theory.

It is not difficult to show that the set of supermaps completely preserving the structure of $\mathcal{M}_N^X$ is not empty. It contains for instance all maps of the product form (22), and also all maps acting on $\mathcal{M}_{N^2}^Q$ which are bistochastic. Any extension of a bistochastic map is bistochastic, and this property guarantees that any initial state majorizes its image, so the structure (11) of the set of $\mathcal{M}_N^X$ is preserved.

On the other hand, the problem of deciding, whether a given map acting in the space of extended states is preserving (completely preserving) is in general not simple, and it is not determined by the (complete) positivity of the map. For instance, a completely positive map which sends all states of $\mathcal{M}_{N^2}^Q$ into a pure state $|\varphi\rangle\langle\varphi|$ (where $|\varphi\rangle \in \mathcal{H}_{N^2}$), is not preserving, since the pure state does not belong to $\mathcal{M}_N^X$, so it cannot be completely preserving. However, the reflection with respect to the maximally mixed state, $\rho' = 2\rho_* - \rho$, is not positive in $\mathcal{M}_4^Q$, but it preserves the smaller set $\mathcal{M}_2^X$ of extended states. It is also known that allowing for a non-product extension followed by a global unitary dynamics and partial trace over the auxiliary subsystem may lead to non-completely positive dynamics [46]. We close the discussion here admitting that the problem of finding an efficient criterion to distinguish the preserving and completely preserving maps remains open.

In order to compare predictions of quartic and quadratic theory we need to find a way to associate with a given supermap, $\Gamma : \mathcal{M}_N^X \to \mathcal{M}_N^X$, a quantum map $\Psi : \mathcal{M}_N^Q \to \mathcal{M}_N^Q$. For a moment let us restrict our attention to completely positive maps which act in the set $\mathcal{M}_N^Q$. It can be represented in the standard Kraus form,

$$\sigma' = \Gamma(\sigma) = \sum_{i=1}^{k} Y_i \sigma Y_i^{\dagger}. \tag{23}$$

The Kraus operators Y_i act now in the extended Hilbert space $\mathcal{H}_{N^2} = \mathcal{H}_A \otimes \mathcal{H}_{A'}$. In contrast to the form (1), which describes all completely positive maps admissible by standard quantum theory, the Kraus form (23) of a supermap provides only a class of measurement processes admissible within the extended quantum theory.

In analogy to (8) we may represent such a supermap Γ by its dynamical matrix

$$G = \Gamma^R = \left(\sum_{i=1}^{k} Y_i \otimes \bar{Y}_i \right)^R. \tag{24}$$

The operators Y_i can be chosen to be orthogonal, so their number k will not be larger than N^4. The Choi matrix G of size N^4 is Hermitian and it acts on $\mathcal{H}_{N^4} = \mathcal{H}_A \otimes \mathcal{H}_{A'} \otimes \mathcal{H}_B \otimes \mathcal{H}_{B'}$. Here label A denotes the principal system, A' its extension which generates the state in the quartic theory, while B and B' represent their counterparts used to apply the Jamiołkowski isomorphism (10).

To simplify the dynamics in the space of extended states it is enough to consider the Choi matrix obtained by normalizing the partial trace of the Choi matrix representing a supermap, $D = (\mathrm{Tr}_{A'B'} G)/N$. The resulting quantum map, $\Phi = D^R$, inherits its properties from the corresponding supermap.

Lemma 3. *Let Γ be a linear supermap acting on $\mathcal{M}_N^X$, so it can be represented by a Hermitian dynamical matrix Γ^R. Construct a quantum map Ψ acting on $\mathcal{M}_N^Q$ by performing the partial trace of the dynamical matrix, $\Psi = D^R$ where $D = \mathrm{Tr}_{A'B'}(\Gamma^R)/N$. If Γ is a completely positive (stochastic, bistochastic) supermap, so is the quantum map Ψ.*

Proof. If a supermap Γ is completely positive, then due to Choi theorem the corresponding dynamical matrix is positive, $G = \Gamma^R \geqslant 0$. So is its partial trace, $ND = \mathrm{Tr}_{A'B'} G$, which implies complete positivity of Ψ. If Γ is trace preserving then $\mathrm{Tr}_{BB'} G = \mathbb{1}_{N^2}$, hence $\mathrm{Tr}_B D = \mathrm{Tr}_{A'} \mathbb{1}_{N^2}/N = \mathbb{1}_N$, which implies trace preserving condition for Ψ. Analogously, if Γ is unital then $\mathrm{Tr}_{AA'} G = \mathbb{1}_{N^2}$ so $\mathrm{Tr}_A D = \mathrm{Tr}_{B'} \mathbb{1}_{N^2}/N = \mathbb{1}_N$ which implies unitality of Ψ. Thus stochasticity (bistochasticity) of the supermap Γ implies the same property of the associated quantum map Ψ. □

If the supermap has a product form, $\Gamma = \Psi_A \otimes \Psi_{A'}$, and all its Kraus operators have the tensor product structure, the corresponding quantum dynamics is given by reduced Kraus operators, $X_i = \mathrm{Tr}_{A'} Y_i$. However, in the case of an arbitrary stochastic Γ this relation does not hold.

In general one may also consider a wider class of supermaps which preserve the set of extended states, but for which the extended Choi matrix $G = \Gamma^R$ is not positive. However we need to require that the induced quantum dynamics is completely positive. This implies a condition for the partial trace

$$D = \mathrm{Tr}_{AA'} G \geqslant 0, \tag{25}$$

which is obviously fulfilled for any positive G. On the other hand relation (25) is satisfied for a large class of operators G which are not positive. This shows that the class of admissible dynamics in the extended theory is wider than in the standard quantum theory.

6. Classical, quantum and extended theories: a comparison

The standard quantum theory reduces to a classical theory if one takes into account only the diagonal parts of a state ρ and restricts the space of operations. Technically, one may define an operation of *coarse graining* with respect to a given Hermitian operator H, which is assumed to be non-degenerate. This operation can be represented as a sum of projectors onto eigenstates $|h_i\rangle$ of H,

$$\rho \;\to\; \Phi_{CG}(\rho) = \sum_{i=1}^{N} |h_i\rangle\langle h_i|\rho|h_i\rangle\langle h_i|. \tag{26}$$

In other words, this map deletes all off-diagonal elements from a density matrix, if represented in the eigenbasis of H and produces a probability vector $\vec{p} = \mathrm{diag}(\rho)$. It consist of N non-negative components, the sum of which is not larger than unity, so $\vec{p}$ lives in the simplex $\widetilde{\mathcal{M}}_N^C = \Delta_{N-1}$. Since off-diagonal elements are called quantum coherences, the process induced by coarse graining is called *decoherence*. The effects of decoherence play a key role in quantum theory and their presence explains why effects of quantum coherence are not easy to register.

In a similar way, for each quantum map Ψ one may obtain reduced, classical dynamics, by taking diagonal elements of dynamical matrix $D = \Psi^R$. The classical transition matrix, $T = [\mathrm{diag}(\Psi^R)]^R$, inherits properties of Ψ, as stated in lemma 1 In particular, if Ψ is a stochastic map, then T forms a stochastic matrix, while if Ψ is a trace non-increasing map, then T is substochastic, what means that the sum of all elements in each its column is not larger than unity.

Consider an arbitrary quantum state ρ, transform it by a stochastic map into $\rho' = \Psi(\rho)$ and perform coarse graining to obtain a classical state $p' = \mathrm{diag}(\rho')$. Alternatively, get the classical vector by coarse graining, $p = \mathrm{diag}(\rho)$, and transform it by reduced dynamics T to arrive at $p'' = Tp$. In general, both vectors are not equal,

$$p'_m = \sum_{ab} \Psi_{\substack{mm\\ab}} \rho_{ab} \neq p''_m = \sum_{abc} \Psi_{\substack{mm\\ab}} \rho_{ac}\delta_{ab}\delta_{ac}, \tag{27}$$

which is a consequence of the known fact that classical and quantum dynamics do differ. Such a direct comparison between discrete classical and quantum dynamics may be succinctly summarized in a non-commutative diagram

$$\begin{array}{llcr} \mathrm{QM}: & \mathcal{M}_N^Q \ni \rho & \xrightarrow{\Psi} & \rho' = \Psi(\rho) \\ \downarrow\ \Phi_{\mathrm{CG}} & \downarrow & \downarrow & \downarrow\ \Phi_{\mathrm{CG}} \\ \mathrm{CM}: & \mathcal{M}_N^C \ni p = \mathrm{diag}(\rho) & \xrightarrow{T(\Psi)} & p'' \neq p' = \mathrm{diag}(\rho') \end{array} \tag{28}$$

Horizontal arrows represent quantum (classical) discrete dynamics, while vertical arrows can be interpreted as the action of the coarse-graining operation defined in equation (26), which reduces quantum theory to classical.

In an analogous way one can compare dynamics with respect to the extended and standard quantum theories. The transition from a state of the quartic theory to a standard quantum mechanical state occurs by taking the partial trace, $\rho = \mathrm{Tr}_{A'}\sigma$. This process can be called a *hyper-decoherence*, since it corresponds to the decoherence which induces the quantum-classical transition. As standard decoherence effects make the observation of the quantum effects difficult, the hyper-decoherence reduces the magnitude of effects unique to the extended theory.

Let us start with an arbitrary state σ of the quartic theory, transform it by an admissible linear supermap map into $\sigma' = \Gamma(\sigma)$ and perform a reduction to obtain the quantum state $\rho' = \mathrm{Tr}_{A'}(\sigma')$. Alternatively, get the quantum state by reduction, $\rho = \mathrm{Tr}_{A'}(\sigma)$, and transform it by reduced quantum map $\Psi(\Gamma)$, characterized in lemma 3, to arrive at $\rho'' = \Psi(\rho)$. In general, both quantum states are different,

$$\rho'_{mn} = \sum_{xyztb} \Gamma_{\substack{mbnb\\xyzt}} \sigma_{\substack{xy\\zt}} \neq \rho''_{mn} = \sum_{xyztb} \Gamma_{\substack{mxny\\ztzt}} \sigma_{\substack{xb\\yb}}. \tag{29}$$

In this way we have justified:

Proposition 2. *The extended quartic theory forms a non-trivial generalization of the standard quantum theory. In particular, there exist experimental schemes (consisting of an initial state and the measurement operators) for which both theories give different predictions concerning probabilities recorded.*

The comparison between dynamics in quartic and quadratic theories is visualized in a non-commutative diagram analogous to (28),

$$\begin{array}{lccc}
\mathrm{XM}: & \mathcal{M}_N^X \ni \sigma & \xrightarrow{\Gamma} & \sigma' = \Gamma(\sigma) \\
\downarrow \ \text{reduction} & \downarrow & \downarrow & \text{partial trace} \downarrow \\
\mathrm{QM}: & \mathcal{M}_N^Q \ni \rho = \mathrm{Tr}_{A'}(\sigma) & \xrightarrow{\Psi(\Gamma)} & \rho'' \neq \rho' = \mathrm{Tr}_{A'}(\sigma')
\end{array} \tag{30}$$

The vertical arrows denote here the operation of partial trace over an auxiliary subsystem and reduction of extended theory to quantum theory, while horizontal arrows represent dynamics in the space of extended (quantum) states.

Thus the classical theory describing dynamics inside the N-dimensional simplex Δ_N of subnormalized probability vectors remains a special case of the quantum theory, in which dynamics takes place in the N^2-dimensional set $\widetilde{\mathcal{M}}_N^Q$ of subnormalized quantum states. In a very similar manner, the standard quantum theory may be considered as a special case of the extended theory, obtained by projecting down the N^4-dimensional set $\widetilde{\mathcal{M}}_N^X$ of extended states into $\widetilde{\mathcal{M}}_N^Q$. A comparison between all three approaches is summarized in table 1. The symbol ^{R} denotes here the transformation of reshuffling of a matrix defined in equation (8).

To reveal similarities between both decoherence processes let us formulate two analogous statements.

Proposition 3 (Decoherence)**.** *A classical state $\vec{p}\,'$ obtained by a decoherence of an arbitrary quantum state corresponding to the classical state $\vec{p}$ satisfies the majorization relation*

$$\vec{p}\,' := \mathrm{diag}(UpU^\dagger) \prec \vec{p}, \tag{31}$$

Table 1. Comparison between classical, quantum and extended theories.

Theory	Classical (linear)	Quantum (quadratic)	Extended (quartic)
Pure states	corners of Δ_{N-1}	$\lvert\psi\rangle \in \mathbb{C}\mathbf{P}^{N-1}$	$U(\lvert\psi\rangle\langle\psi\rvert \otimes \frac{\mathbb{1}}{N})U^\dagger$
Mixed states	$\vec{p} \in \widetilde{\mathcal{M}}_N^C = \Delta_N$	$\rho \in \widetilde{\mathcal{M}}_N^Q$	$\sigma \in \widetilde{\mathcal{M}}_N^X$
Dimensionality	N	N^2	N^4
Dynamics	$\vec{p}' = T\vec{p}$	$\rho' = \Psi(\rho)$	$\sigma' = \Gamma(\sigma)$
Reduction	quantum to classical	extended to quantum	
of the state from	$p = \mathrm{diag}(\rho)$	$\rho = \mathrm{Tr}_{A'}\sigma$	
of the map from	$T = [\mathrm{diag}(\Psi^R)]^R$	$\Psi = [\mathrm{Tr}_{A'B'}\Gamma^R]^R/N$	

where U is a unitary matrix of size N and p stands for a diagonal matrix with vector $\vec{p}$ on the diagonal.

The proof consists in an application of the Schur lemma, which states that the diagonal of a positive Hermitian matrix is majorized by its spectrum. This statement follows also from the Horn–Littlewood–Polya lemma, which says that $\vec{x} \prec \vec{y}$ if there exists a bistochastic matrix B such that $x = By$ (see, e.g. [28]).

Proposition 4 (Hyper-decoherence)**.** *A quantum state ρ' obtained by a hyper-decoherence from an arbitrary extension of a quantum state ρ satisfies the majorization relation*

$$\rho' := \mathrm{Tr}_{A'}[U(\rho \otimes \mathbb{1}/N)U^\dagger] \prec \rho, \tag{32}$$

where U is a unitary matrix of size N^2.

To prove this statement it is enough to observe that the map $\rho' = \Phi(\rho)$ is bistochastic, so according to the quantum analog of the Horn–Littlewood–Polya lemma (see, e.g. [28]), the majorization relation (32) holds. Alternatively, for small system sizes one may use the results of the quantum marginal problem: the inequalities of Bravyi [47] for $N = 2$ and inequalities of Klyachko [48] for $N = 3$ concerning constraints between the spectra of a composite system and its partial traces imply relation (32).

Note that the unitary matrix U is arbitrary, it may in particular represent the swap operation, which exchanges both subsystems. Thus the extended state should not be treated as merely a composition of a 'physical particle' with an 'auxiliary ghost': they are intirinsicly intertwined into a single entity representing an extended state σ, which may be reduced to the standard quantum state ρ' due to the process of hyper-decoherence.

7. Higher order theories

Iterating the extension procedure one can construct higher order theories, in which the number of degrees of freedom scales with dimensionality as $K = N^r$ for any even r. Let us rename the set $\mathcal{M}_N^Q$ of quantum states into $\mathcal{M}_N^{(0)}$, and the set $\mathcal{M}_N^X$ of extended states into $\mathcal{M}_N^{(1)}$. Then we may define the set of states in a mth order generalized theory as in (11),

$$\mathcal{M}_N^{(m)} := \left\{ \sigma \in \mathcal{M}_{N^{1+m}}^{(0)} : \sigma \prec \sigma_0 = |0\rangle\langle 0| \otimes \frac{1}{N^m} \mathbb{1}_{N^m} \right\}, \tag{33}$$

where $|0\rangle \in \mathcal{H}_N$. The parameter $m = 1, 2, 3\ldots$ represents the number of additional ancilliary states: it is equal to zero for the standard quantum theory, $m = 1$ for the extended quartic theory discussed in this work, and $m = 2$ for the next, extended quantum theory for which $K = N^6$. In general, the number K of degrees of freedom behaves as N^{2m+2}, so the exponent r

is equal to $2m+2$. The standard quantum state is obtained by the partial trace over an auxiliary system of size N^m.

The spectra of the states of the higher order theories also form a permutohedron $\mathrm{Perm}(\{N^{-m}\}_{N^m}, \{0\}_{N^m(N-1)})$, defined by the vector of length N^{m+1} containing N^m non-zero equal components. From a geometric point of view increasing the number m of ancillas corresponds to increasing the dimensionality of the Hilbert space and continuing the procedure of truncation of the set of positive operators, which represent quantum states. The larger m, the more faces and corners of the permutohedron which becomes closer to the ball in N^{2m+2} dimensions.

The set of pure states of the higher order theory forms the flag manifold $\mathcal{P}_N^{(m)} = U(N^{m+1})/[(U(N^{m+1} - N) \times U(N)]$ of $2(N^{m+2} - N^2)$ dimensions. The entropy of any state of such a theory belongs to the interval $S(\sigma) \in [m \ln N, (m+1) \ln N]$, so its degree of mixing can be characterized by the gauged entropy, $S_m := S - m \ln N$, which is equal to zero for extended pure states.

As analyzed in section 4 the set of elements of an extended POVM can be defined by the cone dual to the set of extended states, $\mathcal{E}_N^X = (\widetilde{\mathcal{M}}_N^X)^*$. In a similar way, positivity condition analogous to (18) implies that for an extended theory of order m elements of extended POVMs belong to the set

$$\mathcal{E}_N^{(m)} = \left\{E_i = E_i^\dagger : \mathrm{eig}(E_i) \in \left(\mathcal{M}_N^{(m)}\right)^* \text{ and } E_i \leqslant \mathbb{1}_{N^{m+1}}\right\}. \tag{34}$$

The larger m, the larger space one works with and the smaller (more truncated) set of extended states. Accordingly, the larger is the corresponding dual set of elements of extended POVMs.

Discrete dynamics in higher order theories can be defined as in the quartic theory. To get a reduction of a hypermap acting in the space of extended states it is sufficient to perform a suitable partial trace on an extended dynamical matrix and require that the standard Choi matrix D obtained in this way is positive. In this manner one can define an entire hierarchy of hyper-maps which act on the spaces of states of different dimensionalities.

Higher order theories have an appealing feature if the exponent is a power of two, $r = 2^k$. Then one may consider a higher order Jamiołkowski isomorphism: a hypermap acting in the N^r-dimensional space of extended states can be considered as an extended state of the higher order theory characterized by the exponent $2r$. For instance, a supermap Γ of the quartic theory, $k = 2$, may be interpreted as an extended state in an octonic theory for which $k = 3$. The state $\sigma_8 = \rho \otimes \mathbb{1}_{N^3}/N^3 \in \mathcal{M}_N^{(3)}$ describes a supermap Γ_ρ, which sends all quantum maps into Φ_ρ defined in (16).

We have thus shown that the standard quantum mechanics can be embedded in an infinite onion-like structure of higher order theories. Working with a theory characterized by an exponent r one needs N^r parameters to describe the quantum state and to predict the future. Although it is yet uncertain, whether such higher order theories may have a direct physical relevance, investigating how one theory is embedded into another one may contribute to a better understanding of the structure of the standard quantum theory.

8. Concluding remarks

In his 1974 paper on generalized quantum mechanics [29] Mielnik wrote: *the incompleteness of the present day science at this point is, perhaps, one more reason why the scheme of quantum mechanics should not be prematurely closed.* Although more than 30 years have gone, we believe that this statement is still valid and it provides motivation for the present work.

Making use of a geometric approach to quantum mechanics we considered different constructions of the arena for an extended quantum theory. In particular for the case of mono-partite systems we have formulated a generalization of the standard quantum theory for which the number of degrees of freedom scales as the fourth power of the number of distinguishable states. As the standard quantum theory is somehow coded in the shape of the $N^2 - 1$-dimensional convex set $\mathcal{M}_N^Q$ of all normalized states acting on the N-dimensional Hilbert space, the same is true for the extended theory which deals with $N^4 - 1$-dimensional set convex $\mathcal{M}_N^X$ of extended states. It is tempting to believe that the present paper allows one to formulate a long-term research project on plausible generalizations of the quantum theory and its interdisciplinary implications.

The extended theory proposed in this work is based on the following steps:

(i) *extended states* belong to the set $\mathcal{M}_N^X$ which forms a part of the set $\mathcal{M}_{N^2}^Q$ of the states of a bipartite system of the standard quantum theory truncated in such a way to support N distinguishable states.

(ii) *extended measurements* are formed by elements E_i of the set $\mathcal{E}_N^X$ dual to $\mathcal{M}_N^X$.

(iii) *extended dynamics* is described by linear maps preserving the set of extended states for which the reduced dynamical matrix is positive (25).

The states of the extended quartic theory admit a dynamical interpretation in view of the Jamiołkowski isomorphism [37]. Any extention $\sigma \in \mathcal{M}_N^X$ of the quantum state $\rho \in \mathcal{M}_N^Q$ represents a completely positive map $\Phi : \mathcal{M}_N^Q \to \mathcal{M}_N^Q$, which sends the maximally mixed state into the state in question, $\Phi(\mathbb{1}/N) = \rho$. Geometric approach explored in this work allows us to construct an infinite hierarchy of higher order theories and to extend the Jamiołkowski isomorphism: any state of the theory of order $k+1$ can be interpreted as a map acting on the set of states defined in the theory of order k.

The extended quartic theory includes the quadratic quantum theory as a special case: if the initial state and the measurement operators have the tensor product form both theories yield exactly the same results. More formally one can assume that the entire system in the framework of the extended theory can be approximated by the Hamiltonian

$$H \approx H_{\rm phys} + H_{\rm gh} + \lambda H_{\rm int}, \tag{35}$$

in which $H_{\rm phys}$ and $H_{\rm gh}$ represent the 'physical particle' and the ancilliary subsystem, respectively, while $H_{\rm int}$ describes the interaction between them. The effective coupling constant λ can be then defined by the ratio of the expectation values, $\langle H_{\rm int}\rangle/\langle H_{\rm phys}\rangle$. The correspondence principle is obtained taking the limit $\lambda \to 0$, since in this case the presence of the ancilliary system will not influence the physical system. On the other hand, the quartic theory is more general than the standard quantum theory and for a non-zero coupling constant λ it may give different predictions for the probabilities of outcomes of certain measurements.

A reader will note that the structure of composite systems is not touched in this scheme, which right now concerns simple systems only. Generalization of the extended theory for composite systems turns out not to be simple. For instance, accepting the tensor product structure used in quantum theory, one faces the problem that the same mathematical notion of partial trace has to be used to trace out the fictitious 'ghost particle' of an extension or a part of the physical subsystem. In particular, an extended state $\rho_{ABA'B'}$ of two qubits, reduced by the partial trace with respect to the ancillas $A'B'$ can form an arbitrary two-qubit state $\rho_{AB} = \mathrm{tr}_{A'B'}(\rho_{ABA'B'}) \in \mathcal{M}_{16}^Q$, while the same state reduced by one subsystem gives an extended state $\rho_{AA'} = \mathrm{tr}_{BB'}(\rho_{ABA'B'})$ which is required to belong to a smaller set $\mathcal{M}_2^X \subset \mathcal{M}_4^Q$. This may suggest that the extended state of two qubits $\mathcal{M}_{2,2}^X \in \mathcal{M}_{16}^Q$ cannot be invariant with respect to the unitary group $U(16)$, so the standard tensor product rule, used in quantum mechanics to describe composed systems, cannot be directly adopted for the extended quartic theory.

The question whether the extended theory proposed here may be generalized for composite systems and might describe the physical reality remains open. Let us mention here that composite systems also pose a difficulty in nonlinear generalizations of quantum theory. On the other hand, it is also thinkable to treat the entire universe as the only physical system, and to agree that splitting it into various subsystems is performed for practical purposes as a kind of an inevitable approximation.

Even concerning single-particle systems, it remains as a challenge to propose an experimental scheme in which predictions of the extended (quartic) theory constructed in this work would differ significantly from results following from the standard (quadratic) quantum theory. The key issue is to find a way to prepare a non-standard initial state σ which is not in a product state but reveals an entanglement with the ancilla. A suitably chosen detection scheme could then reveal usefulness of the generalized quartic mechanics, and detect the presence of the hypothetical ancilliary state associated with this theory.

If this step turns out not to be realistic, by analyzing experimental data one can try to get an upper bound for the time of the hyper-decoherence: it is thinkable that in standard experimental conditions such hyper-decoherence effects occur so fast that fine effects due to correlations with the auxiliary system cannot be observed. However, this could be the case for systems in extremal conditions like these characterized by very high energies or ultrastrong fields and important for the investigations of the early history of the universe.

Construction of an extended, quartic theory might have consequences for the axiomatic approach to quantum mechanics. Since the proposed set-up does not include composite systems, the present work does not imply that the last, simplicity axiom proposed by Hardy [10] is inevitable to derive the standard quantum theory in an axiomatic way. It is also plausible that this axiom is not necessary, if the quantum theory is the only theory which satisfies all other axioms including the one on the description of composite systems.

Our work provides a starting point in an attempt to construct a fruitful generalization of the standard quantum theory and leaves many questions open. One could look for a suitable algebraic viewpoint to study properties of the quartic (higher order) quantum theory or to investigate possibilities of constructing an extended quantum theory of fields. For instance, the 'ghost-like' ancillary subsystem, used to interpret the extended quartic theory, could correspond in the path integration approach to extending the path of integration by two additional 'ghost-like' points, which vanish in the standard theory. Alternatively, one could examine, if such an extended quantum theory is related to the 'thermo field dynamics' of Umezawa *et al* in which a 'thermal vacuum state' is introduced [49].

Furthermore, it would be interesting to study a possible link between the higher order theory and the generalized quantum mechanics of Sorkin to verify, whether the higher order interference terms, present in the generalized measure theory [24, 50], are related to the extended quantum theory. Although the quartic quantum theory is linear, following the approach of Mielnik [29] one could also analyze possible relations to certain nonlinear generalizations of quantum theory, see, e.g. [51–53].

The theory of information processing can be studied not only within the classical or the quantum set-up, but also in more general probabilistic theories [25, 40, 54–58]. Hence one could also attempt to analyze implications of the extended quartic theory for the quantum information processing. During the last two decades it has been investigated to what extend the transition from classical bits to quantum qubits gives additional possibilities for information processing. In a similar way, one could study consequences of a further enlargement of the scene for an information processing screenplay.

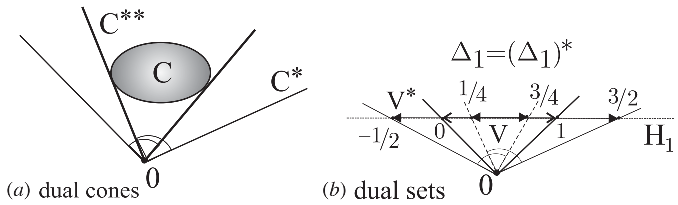


Figure 5. Dual cones and sets: (*a*) a set C, its dual cone C^*, and the cone $C^{**} \supset C$; (*b*) a set $V_a \subset \Delta_1$ and its dual set $V_a^* \subset H_1$ plotted here for $a = 1/4$.

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Appendix A. Dual cones and dual sets

In this appendix we provide necessary concepts of convex analysis and prove a lemma on dual sets.

Consider any set C in $\mathbb{R}^n$. Its *dual cone* is defined as [59]

$$C^* := \{y \in \mathbb{R}^n : x \cdot y \geqslant 0, \forall x \in C\}. \tag{A.1}$$

The set $C^{**} = (C^*)^*$ is the closure of the smallest cone containing C. A cone C is said to be *self-dual* if $C = C^*$. The non-negative orthant of $\mathbb{R}^n$ and the space of all positive semi-definite matrices are self-dual.

Normalization condition $\sum_{i=1}^n x_i = 1$ defines a hyperplane H_1 in $\mathbb{R}^n$. Let V be an arbitrary set obtained as a cross-section of a convex cone C, with the hyperplane, $V = C \cap H_1$. Then its *dual set* is given by the cross-section of the hyperplane with the dual cone, $V^* = C^* \cap H_1$ (see figure 5). Note that the probability simplex is self-dual, $\Delta_N = \Delta_N^*$. Consider, for instance, a symmetric subset of the $N = 2$ simplex, $V_a = [a, 1 - a]$ with $a \in [0, 1/2]$. Then the dual set reads $V_a^* = [b, 1 - b]$ where $b = a/(2a - 1)$. Hence if V reduces to a single point for $a = 1/2$, its dual V^* covers entire line. Two more examples of pairs of dual sets, which live in the plane H_1 defined by the condition $x_1 + x_2 + x_3 = 1$ are shown in figure 6.

The same concept of dual sets can be applied for the set of Hermitian operators. Instead of the standard scalar product one uses in this case the Hilbert–Schmidt product, $\langle A|B\rangle = \mathrm{tr}\, A^\dagger B$, while the hyperplane is introduced by the trace normalization, $A \in H_1 \Leftrightarrow A = A^\dagger, \mathrm{tr}\, A = 1$. Hence for any set V in H_1 its dual reads

$$V^* := \{A \in H_1 : \mathrm{tr}\, AB \geqslant 0, \forall B \in V\}. \tag{A.2}$$

Taking a brief look at equation (18) we see that the cone containing the set $\mathcal{M}_N$ of extended states and the cone $\mathcal{E}_N$ of admissible elements of a POVM have to be dual. In the standard quantum theory (quadratic), these cones are self-dual, $\left(\mathcal{M}_N^Q\right)^* = \mathcal{E}_N^Q$. However, this

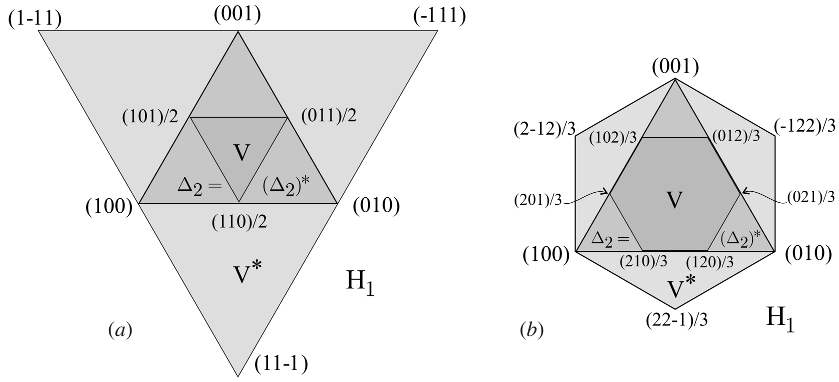

Figure 6. Self-dual simplex $\Delta_2 = \Delta_2^*$ of $N = 3$ classical states living in the plane H_1 and its subsets: (*a*) triangle $V = \text{Perm}(1/2, 1/2, 0)$ and its dual triangle $V^* = \text{Perm}(1, 1, -1)$; (*b*) hexagon $V = \text{Perm}(2/3, 1/3, 0)$ and its dual hexagon $V^* = \text{conv hull}(\text{Perm}(2/3, 2/3, -1/3) \cup \text{Perm}(1, 0, 0))$.

is not the case for extended (quartic) theory: since $\mathcal{M}_N^X$ is obtained by *truncating* the set of positive operators as discussed in section 4, its dual cone $\left(\mathcal{X}_N^X\right)^*$ is *extended* to contain also some non-positive operators.

The structure of the set dual to some set of Hermitian matrices gets simpler if we consider a class of sets, which are invariant with respect to all unitary operations, $A \in V_Q \Rightarrow UAU^\dagger \in V_Q$. In such a case one can reduce the problem of finding the dual set X^* of operators to a simpler problem of finding the set dual to the set of all admissible spectra.

Lemma 4. *Let V be a convex subset of the simplex Δ_{N-1} of classical probability vectors of length N, which is invariant with respect to all $N!$ permutations of the vector components. Let V_Q denote the set of all Hermitian matrices $UvU^\dagger$, where U is unitary and v is a diagonal matrix of size N such that $\text{diag}(v) \in V$. Then the dual set $(V_Q)^*$ contains all Hermitian operators with spectra belonging to V^*, so it can be called $(V^*)_Q$.*

Observe that this lemma can also be visualized by figure 6: if two sets of classical normalized vectors V and V^* from H_1 are dual, so are the sets V_Q and $(V^*)_Q$ of Hermitian operators with spectra belonging to V and V^*, respectively. Before presenting its proof let us quote a related lemma proved in [43].

Lemma 5. *Consider a state $\rho \geqslant 0$ with spectrum $\vec{p}$ and a Hermitian operator $\sigma = \sigma^\dagger$ not necessarily positive, with spectrum $\vec{q}$. Then their trace is bounded as*

$$p^\uparrow \cdot q^\downarrow \leqslant \text{Tr}\,\rho\sigma \leqslant p^\uparrow \cdot q^\uparrow, \tag{A.3}$$

where $p^\uparrow$ $(p^\downarrow)$ denotes vectors in an increasing (decreasing) order.

Proof of lemma 4. Lemma 5 applied to a positive operator $A \in V_Q$ and a Hermitian operator $B = B^\dagger$ with spectrum $\vec{q}$ gives a lower bound for the trace $\text{tr}\,AB$ in terms of their ordered spectra. Since $\vec{p} = \text{eig}(A) \in V$ so that if $\vec{q}$ belongs to the dual set V^* their scalar product $\vec{p} \cdot \vec{q}$ is non-negative. This relation holds for any order of the components of both vectors, since V is invariant with respect to permutations. This fact implies in turn that $\text{tr}\,AB \geqslant 0$ for any Hermitian operator B with spectrum $\vec{q}$, which is equivalent to the statement $B \in (V^*)_Q$. In this way we have shown that the set dual to V_Q is equal to $(V^*)_Q$. □

In particular lemma 4 implies relation (20) which shows that the set $\mathcal{E}_N^X$ of elements of XPOVM contains Hermitian operators with spectra from the cone dual to the set of spectra of all extended states $\mathcal{M}_N^X$. Furthermore, for any extended theory of oder m it follows that relation (34) holds and the cones of extended states and elements of extended POVMs are dual.

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